The laminar boundary layer on oscillating plates and cylinders

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SUMMARY

This paper considers the two-dimensional laminar boundary layer on an infinite flat plate normal to an oncoming stream, the plate making transverse oscillations in its own plane. The exact solution is shown to depend on a single ordinary differential equation, containing the frequency of oscillation as a parameter. Series methods are employed to evaluate the solution for small and large values of the frequency, and enough terms are calculated to give the solution with satisfactory accuracy over the whole frequency range. It is observed that the solution satisfies the full Navier–Stokes equations.

For a cylinder of arbitrary section making transverse or rotational oscillations, it is shown how the results for a flat plate can be used to describe the boundary layer in the neighbourhood of the front stagnation point. Difficulties which arise in extending the solution to cover the remainder of the cylinder are discussed, and an estimate is made of the fluctuating torque on a circular cylinder making transverse oscillations.

1. INTRODUCTION

Attention has recently been paid to the laminar boundary layer in flows in which there are fluctuations superposed on a basic steady low-speed motion. With the two-dimensional flow about an infinite cylinder as the basic motion, Wuest (1952) has considered the effect of oscillations of the cylinder in the axial direction, and Lighthill (1954) has considered oscillations parallel to the oncoming stream. However, in flutter problems, rotational or transverse oscillations are of chief importance, and it is with these cases that this paper is concerned. From the point of view of the boundary layer, it is immaterial whether it is the cylinder or the oncoming stream which is oscillating, as the inertial effects of an acceleration applied to the whole system are countered by a uniform pressure gradient, and the relative motion is completely unaffected. Compressibility effects will not modify this conclusion provided that the acoustic wave-length is large compared with the dimensions of the cylinder section.

The results of this paper all follow from a detailed analysis of the two-dimensional flow against an infinite flat plate normal to the stream, the plate making transverse oscillations in its own plane. In § 2 it is shown that for all amplitudes and frequencies of oscillation, the perturbation of the flow is given by a single ordinary differential equation, containing the frequency as a parameter. Series expansions are obtained, valid for small and large frequencies, and enough terms are calculated to give the solution with satisfactory accuracy over the whole frequency range. Further, the solution satisfies not only the boundary layer equations, but also the full Navier–Stokes equations.

At the lowest frequencies, the problem is identical with that of a steadily moving plate, and the perturbation to the boundary layer profile is determined simply as the derivative of the basic profile, the well-known Hiemenz function. At the highest frequencies, the perturbation is a shear-layer, exactly as on a plate oscillating in a fluid at rest. It is found that the phase of the oscillating component of the skin-friction is always in advance of the velocity fluctuation, the phase-advance being proportional to the frequency for low frequencies, and approaching the limit $\frac{1}{4}\pi$ for high frequencies.

This behaviour is similar to that found by Lighthill (1954) for oscillations in the magnitude of the oncoming stream. Although he had available only the first term in each of the series for large and small frequencies, he was able to show that these are sufficient to cover the whole frequency range in that case. Stuart (1955) has considered the case of a fluctuating stream parallel to an infinite porous plate, and has obtained a solution which is valid for all frequencies. The phase-advance again has the same qualitative behaviour, but, as in the present case, the first terms in the low and high frequency approximations are not themselves sufficient to cover the whole frequency range. In the case of axial oscillations of a cylinder considered by Wuest (1952), it is well-known that the axial motion has no effect on the flow in planes normal to the axis, and that the axial flow satisfies a linear differential equation. As well as considering more general cases, Wuest solved numerically the equation for the flow near the stagnation point for one particular frequency. It may be noted that, for general values of the frequency, the solution could be found by a straightforward application of the methods of the present paper. It is clear that the variation of phase-advance with frequency is similar to that in the other cases which have been treated.

In § 3, it is shown how the results of § 2 may be applied to the boundary layer near the stagnation point on a cylinder making transverse or rotational oscillations. The first obstacle to be overcome in a consideration of the boundary layer further downstream on the cylinder is the difficulty of specifying the form of the fluctuating component of the velocity distribution at the edge of the boundary layer. In the case of streamwise oscillations, Lighthill was able to assume that the velocity distribution outside the boundary layer remained unchanged in form, and he succeeded in developing a very satisfactory Kármán–Pohlhausen treatment of the boundary layer on a cylinder of arbitrary section. In the present case, it may perhaps be hoped that the considerations of § 3 will serve as a starting point for a Kármán–Pohlhausen attack on the problem, and enable the solution to be continued away from the awkward region around the fluctuating stagnation point.

2. Oscillating plate

The problem to be considered in this section is that of the two-dimensional flow of an incompressible fluid against an infinite plate normal to the stream. The theory will be developed for the case in which the plate makes harmonic oscillations in its own plane, and it will then be shown how the results can be immediately applied to the case where the plate is fixed but the point to which the dividing streamline in the oncoming flow is directed oscillates.

Take Cartesian coordinates (x, y) fixed in space, the x-axis being along the plate and the y-axis normal to it, so that $x=0$ is the dividing streamline in the steady flow outside the boundary layer on the plate. Thus, if (u, v) are the corresponding components, outside the boundary layer $u=U=cx$, where c is a constant, and so, by continuity, $v=V=-c(y-\delta)$. The value of δ will be determined in the course of the solution, and found to be constant. U and V are seen to be the velocity components in the two-dimensional potential flow against the plane $y=\delta$. The plate is assumed to have velocity $ae^{i\omega t}$ in the x-direction, where a and ω are constants.

The boundary layer equations are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2} \tag{1}$$

and

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2}$$

where ν is the kinematic viscosity. These are to be solved with the boundary conditions

$$u=ae^{i\omega t},\quad v=0,\quad \text{at } y=0,\quad \text{and}\quad u\to U=cx\quad \text{as } y\to\infty. \tag{3}$$

For $a=0$, the plate is fixed and the solution takes the classical form obtained by Hiemenz in which the velocity components are $u=cxf'(\eta)$, $v=-(c\nu)^{1/2}f(\eta)$, where $\eta=(c/\nu)^{1/2}y$, a dash denotes differentiation with respect to η, and f satisfies the equation

$$f'''+ff''-f'^2+1=0 \tag{4}$$

with boundary conditions $f(0)=0$, $f'(0)=0$, $f'(\infty)=1$. Equation (4) has been studied in detail by many writers. A ten-figure tabulation of f, f' and f'', performed by Dr N. E. Hoskin on the Manchester electronic computer, proved invaluable during the subsequent calculations of this paper.

To satisfy equation (3), we look for a solution in the form

$$u=cxf'(\eta)+ae^{i\omega t}\phi(\eta),\quad v=-(c\nu)^{1/2}f(\eta), \tag{5}$$

where the meanings of f and η are unchanged. Since η is a function of y only, the continuity equation (2) is still satisfied. Equation (1) now becomes

$$c^2x\{f'''+ff''-f'^2+1\}+ace^{i\omega t}\{\phi''+f\phi'-f'\phi-\frac{i\omega}{c}\phi\}=0. \tag{6}$$

The first bracket is zero, by equation (4), so both the equation and the boundary conditions (3) will be satisfied provided

$$\phi''+f\phi'-f'\phi=\frac{i\omega}{c}\phi;\quad \phi(0)=1,\quad \phi(\infty)=0. \tag{7}$$

Thus, as for a fixed plate, there is only one ordinary differential equation to be solved. It is to be noted that this result holds for all values of ω and a, so there are no restrictions on the amplitude or frequency of the oscillation. Further, this solution, like the Hiemenz solution itself, does in fact satisfy the full Navier–Stokes equations and not just the boundary layer equations. The only term omitted in the momentum equation (1) is $\partial^2u/\partial x^2$, which vanishes, and, since there is no extra contribution to v, the momentum equation in the y-direction is unaffected.

Since $f(\eta)\sim\eta-0{\cdot}6479$ as $\eta\to\infty$, the potential flow outside the boundary layer is that against the plane $y=0{\cdot}6479(\nu/c)^{1/2}$.

The only parameter in equation (7) is the frequency ratio ω/c. Two series solutions will be developed, valid respectively for small and large values of ω/c, and it will be shown that sufficient terms are here calculated to enable the whole frequency range to be covered satisfactorily, so far as the value of the skin-friction is concerned.

Small values of ω/c.

Consider first the case $\omega=0$, which implies that the plate velocity has the constant value a. Then, by equation (7), $\phi=\phi_0$, where

$$\phi_0''+f\phi_0'-f'\phi_0=0;\quad \phi_0(0)=1,\quad \phi_0(\infty)=0. \tag{8}$$

Now, differentiation of equation (4) gives

$$ff^{\text{iv}}+ff'''-f'f''=0, \tag{9}$$

and so $\phi_0=f''$ satisfies equation (8). Also $f''(0)=A=1{\cdot}2326$, a value found by the integration of equation (4), and $f''(\infty)=0$. Hence

$$\phi_0=\frac{1}{A}f''=0{\cdot}8113f'' \tag{10}$$

satisfies both the boundary conditions. Thus, in the flow against a plate moving with steady velocity a, the velocity components have the simple form

$$u=cxf'(\eta)+0{\cdot}8113af''(\eta),\quad v=-(c\nu)^{1/2}f(\eta). \tag{11}$$

For small but non-zero values of ω/c, write

$$\phi(\eta)=\sum_{n=0}^{\infty}\left(\frac{i\omega}{c}\right)^n\phi_n(\eta). \tag{12}$$

Consideration of the terms in $(i\omega/c)^n$ in equation (7) shows that, for $n\geqslant 1$,

$$\phi_n''+f\phi_n'-f'\phi_n=\phi_{n-1};\quad \phi_n(0)=0,\quad \phi_n(\infty)=0. \tag{13}$$

Equation (13), like equation (8), has f'' as one integral belonging to its complementary function. The other integral is easily found to be $f''\int_0^\eta \frac{e^{-f^0}}{f''^2}\,d\eta$, where $f^0=\int_0^\eta f\,d\eta$. The method of variation of parameters then leads to the solution

$$\phi_n = -f''\int_0^\eta f''e^{f^0}I\phi_{n-1}\,d\eta - f''I\int_\eta^\infty f''e^{f^0}\phi_{n-1}\,d\eta, \tag{14}$$

where

$$I=\int_0^\eta \frac{e^{-f^0}}{f''^2}\,d\eta.$$

Further details of this solution are given in the appendix. It is shown that the boundary conditions are satisfied, and that ϕ_n tends to zero exponentially as η tends to infinity, for all n.

Actually, ϕ_1 can be obtained directly in a rather simpler manner. The equation for ϕ_1 is

$$\phi_1''+f\phi_1'-f'\phi_1=\phi_0=\frac{1}{A}f''; \quad \phi_1(0)=0, \quad \phi_1(\infty)=0. \tag{15}$$

A particular integral is seen to be $\phi_1=\frac{1}{A}f$. To this must be added suitable multiples of the complementary functions f'' and $f''I$ so as to satisfy the boundary conditions. Equation (56) of the appendix shows that the required solution is

$$\phi_1 = 0{\cdot}8113f - 0{\cdot}6078f''I. \tag{16}$$

This expression can also be obtained from equation (14), with the help of equation (9) and its derivative.

Numerically, we shall confine our attention to the value of the skin-friction, equal to $\mu(\partial u/\partial y)_{y=0}$. The oscillating component of the skin-friction τ is given, from equation (5), by

$$\frac{\tau}{\rho a^2}=\left(\frac{c\nu}{a^2}\right)^{1/2} e^{i\omega t}\phi'(0). \tag{17}$$

The contributions to $\phi'(0)$ from ϕ_0 and ϕ_1 are immediately available, since it is known that $f''(0)=A$, $f'''(0)=-1$, while that from ϕ_2 require only the computation of $\int_0^\infty f''e^{f^0}\phi_1\,d\eta$. Numerical integration yielded the value $-0{\cdot}1167$, so that the first three terms of the series give

$$-\phi'(0)=0{\cdot}8113+0{\cdot}4932\frac{i\omega}{c}+0{\cdot}0947\frac{\omega^2}{c^2}. \tag{18}$$

(It is natural to consider $-\phi'(0)$, since a negative value of the skin-friction would be expected to be associated with a positive velocity of the plate.)

Large values of ω/c.

When ω/c is large, it is convenient to change the variable in equation (7) from η to

$$Y=(i\omega/c)^{1/2}\eta=(i\omega/\nu)^{1/2}y. \tag{19}$$

Write $(c/i\omega)^{1/2}=\alpha$; then, since $\dfrac{d}{d\eta}=\dfrac{d}{\alpha dY}$, equation (7) becomes

$$\phi^{**}-\phi=\alpha\{f^{*}\phi-f\phi^{*}\};\quad \phi(0)=1,\quad \phi(\infty)=0, \tag{20}$$

where a star denotes a derivative with respect to Y.
The expansion for $f(\eta)$ near $\eta=0$ is

$$\begin{aligned} f&=\frac{1}{2}A\eta^2-\frac{1}{6}\eta^3+\frac{1}{120}A^2\eta^5-\frac{1}{360}A\eta^6+\frac{1}{2520}\eta^7+\dots \\ &=\frac{1}{2}A\alpha^2Y^2-\frac{1}{6}\alpha^3Y^3+\frac{1}{120}A^2\alpha^5Y^5-\frac{1}{360}A\alpha^6Y^6+\frac{1}{2520}\alpha^7Y^7+. \end{aligned} \tag{21}$$

For large frequency, α is small, so we look for a solution in series by writing

$$\phi=\sum_{n=0}^{\infty}\alpha^n\varphi_n(Y), \tag{22}$$

and equating to zero the coefficients of successive powers of α in equation (20). The boundary conditions are

$$\varphi_0(0)=1,\quad \varphi_n(0)=0\quad (n\geqslant 1),\quad \varphi_n(\infty)=0\quad (\text{all } n). \tag{23}$$

The equation for φ_0 is

$$\varphi_0^{**}-\varphi_0=0, \tag{24}$$

and the required solution is

$$\varphi_0=e^{-Y}. \tag{25}$$

This is the familiar shear-wave solution due to Stokes, which gives the flow due to a plate oscillating in its own plane in a fluid at rest.

The second and third equations show that φ_1 and φ_2 are zero. The next two equations are

$$\varphi_3^{**}-\varphi_3=Ae^{-Y}\{Y+\tfrac{1}{2}Y^2\} \tag{26}$$

and

$$\varphi_4^{**}-\varphi_4=-e^{-Y}\{\tfrac{1}{2}Y^2+\tfrac{1}{6}Y^3\}. \tag{27}$$

Now, when

$$\varphi=e^{-Y}\sum_{k=0}^{\infty}a_kY^k,$$

$$\varphi^{**}-\varphi=e^{-Y}\sum_{k=0}^{\infty}\{(k+2)(k+1)a_{k+2}-2(k+1)a_{k+1}\}Y^k, \tag{28}$$

so the solutions of (26) and (27) satisfying the boundary conditions (23) are

$$\varphi_3=-Ae^{-Y}\left\{\frac{3}{8}Y+\frac{3}{8}Y^2+\frac{1}{12}Y^3\right\} \tag{29}$$

and

$$\varphi_4=e^{-Y}\left\{\frac{3}{16}Y+\frac{3}{16}Y^2+\frac{1}{8}Y^3+\frac{1}{48}Y^4\right\}. \tag{30}$$

The function φ_5 is identically zero. In the equations for φ_6, φ_7 and φ_8, there are contributions to the right-hand side from φ_3 and φ_4 as well as from φ_0. Making use of equation (28), we finally obtain

$$\varphi_6 = A^2 e^{-Y}\left\{\frac{33}{128}Y + \frac{33}{128}Y^2 + \frac{11}{64}Y^3 + \frac{9}{128}Y^4 + \frac{3}{160}Y^5 + \frac{1}{360}Y^6\right\}, \quad (31)$$

$$\varphi_7 = -Ae^{-Y}\left\{\frac{129}{256}Y + \frac{129}{256}Y^2 + \frac{43}{128}Y^3 + \frac{41}{256}Y^4 + \frac{103}{1920}Y^5 + \frac{73}{5760}Y^6 + \frac{31}{20160}Y^7\right\}, \quad (32)$$

$$\varphi_8 = e^{-Y}\left\{\frac{139}{512}Y + \frac{139}{512}Y^2 + \frac{139}{768}Y^3 + \frac{139}{1536}Y^4 + \frac{127}{3840}Y^5 + \frac{103}{11520}Y^6 + \frac{73}{40320}Y^7 + \frac{31}{161280}Y^8\right\}. \quad (33)$$

Since

$$\phi'(0) = \frac{1}{\alpha}\phi^*(0) = \sum_{n=0}^{\infty} \alpha^{n-1}\varphi_n^*(0),$$

the contribution to the skin-friction from the calculated terms is given by

$$-\phi'(0) = \frac{1}{\alpha} + \frac{3}{8}A\alpha^2 - \frac{3}{16}\alpha^3 - \frac{33}{128}A^2\alpha^5 + \frac{129}{256}A\alpha^6 - \frac{139}{512}\alpha^7$$

$$= \left\{\frac{1}{\sqrt{2}}\left(\frac{c}{\omega}\right)^{-1/2} + \frac{3\sqrt{2}}{32}\left(\frac{c}{\omega}\right)^{3/2} + \frac{33\sqrt{2}}{256}A^2\left(\frac{c}{\omega}\right)^{5/2} - \frac{139\sqrt{2}}{1024}\left(\frac{c}{\omega}\right)^{7/2}\right\}$$

$$+ i\left\{\frac{1}{\sqrt{2}}\left(\frac{c}{\omega}\right)^{-1/2} - \frac{3}{8}A\frac{c}{\omega} + \frac{3\sqrt{2}}{32}\left(\frac{c}{\omega}\right)^{3/2} - \frac{33\sqrt{2}}{256}A^2\left(\frac{c}{\omega}\right)^{5/2} + \frac{129}{256}A\left(\frac{c}{\omega}\right)^3 - \frac{139\sqrt{2}}{1024}\left(\frac{c}{\omega}\right)^{7/2}\right\}, \quad (34)$$

since $\quad \alpha = (c/i\omega)^{1/2} = (1-i)(c/2\omega)^{1/2}.$

The variation of the real and imaginary parts of $-\phi'(0)$ with frequency, as given by (34) for high frequencies and (18) for low frequencies, is shown in figure 1. The close agreement between the two formulae in the region near $\omega/c = 1$ in each case affords good evidence that the terms calculated are sufficient to predict a reliable value of the skin-friction over the whole frequency range. The fact that the real and imaginary parts of $-\phi'(0)$ have the same sign indicates that, in all cases, the skin-friction is advanced in phase relative to the velocity fluctuation. The variation of the phase-advance angle θ_ϕ, which is the argument of $-\phi'(0)$, with frequency is also shown in figure 1. θ_ϕ rises steadily from 0 to $\pi/4$, as the frequency increases. For small frequencies, $\theta_\phi = 0{\cdot}6078\,\omega/c + O(\omega^3/c^3)$, indicating a time of anticipation $\theta_\phi/\omega = 0{\cdot}6078/c$, which is independent of frequency as long as ω^2/c^2 is negligible.

When the plate velocity is not sinusoidal, but may be expressed as a sum of sinusoidal components, it is clear from the form of equations (5) and (6) that the combined effect is the sum of the effects of the individual components.

In particular, if the frequencies of all the components lie in the range for which ω^2/c^2 is negligible, the time of anticipation in the fluctuating component of the skin-friction remains as $0{\cdot}6078/c$.

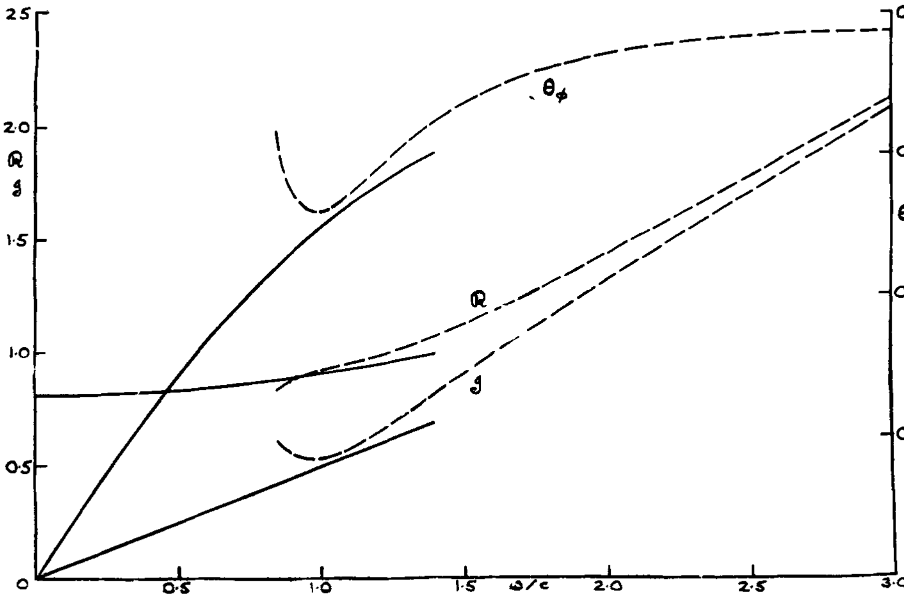

Figure 1. Oscillating plate. Variation with frequency of the real part ($\mathscr{R}$) and the imaginary part ($\mathscr{I}$) of $-\phi'(0)$, and of the phase-advance angle θ_ϕ.

Oscillating stream.

When the dividing streamline of the oncoming stream oscillates in position, but the plate is at rest, the situation differs from that already considered only by the superposition of a uniform, though not constant, transverse velocity, which has no effect on the relative motion. It follows that the solution obtained above can be applied at once to this new case, though the details require a little care.

With coordinates as before, the velocity just outside the boundary layer is

$$U = c\left(x + \frac{b}{c}e^{i\omega t}\right) \tag{35}$$

The stagnation streamline is thus instantaneously directed towards $x = -(b/c)e^{i\omega t}$. The plate itself is at rest.

Take a new origin of coordinates O_1 at $x = de^{i\omega t}$, and write $x_1 = x - de^{i\omega t}$ as the new coordinate along the plate. The point O_1 has velocity $i\omega de^{i\omega t}$, so the stream velocity at x, relative to the new axes, is

$$U_1 = cx + be^{i\omega t} - i\omega de^{i\omega t} = cx_1 + (b + cd - i\omega d)e^{i\omega t}. \tag{36}$$

Now d must be chosen so that U_1 does not fluctuate; hence

$$b + d(c - i\omega) = 0. \tag{37}$$

In the new axes, the plate velocity is $-i\omega d e^{i\omega t}$, and on writing

$$a = -i\omega d \tag{38}$$

we recover the case of the oscillating plate precisely as treated above.

The velocity distribution is

$$u_1 = c x_1 f'(\eta) + a e^{i\omega t}\phi(\eta). \tag{39}$$

On expressing x_1 and a in terms of x and b, we obtain the velocity in the original coordinates as

$$u = u_1 + i\omega d e^{i\omega t} = c x f'(\eta) + b e^{i\omega t}\chi(\eta), \tag{40}$$

where

$$\chi(\eta) = \frac{f' - \dfrac{i\omega}{c} + \dfrac{i\omega}{c}\phi}{1 - \dfrac{i\omega}{c}}. \tag{41}$$

Alternatively, it may be easily shown directly from the boundary layer equations that χ satisfies

$$\chi'' + f\chi' - f'\chi = \frac{i\omega}{c}\chi - 1 - \frac{i\omega}{c}; \quad \chi(0) = 0, \quad \chi(\infty) = 1. \tag{42}$$

Equation (42) is more troublesome to consider directly than was equation (7). But it is easy to check that the value of χ given by (41) does satisfy equation (42) for all values of ω/c.

The result (42) is again exact, with no restrictions as to amplitude or frequency.

For small values of ω/c, since

$$\phi = \phi_0 + \frac{i\omega}{c}\phi_1 - \frac{\omega^2}{c^2}\phi_2 + \dots,$$

equation (41) gives

$$\chi = f' + \frac{i\omega}{c}(f' - 1 + \phi_0) - \frac{\omega^2}{c^2}(f' - 1 + \phi_0 + \phi_1) - \frac{i\omega^3}{c^3}(f' - 1 + \phi_0 + \phi_1 + \phi_2) + \dots, \tag{43}$$

and for large values of ω/c, since

$$\phi = \varphi_0 + \alpha^3\varphi_3 + \alpha^4\varphi_4 + \dots,$$

$$\chi = (1 - \varphi_0) + \alpha^2(1 - f' - \varphi_0) + \alpha^3(-\varphi_3) + \alpha^4(1 - f' - \varphi_0 - \varphi_4) + \dots. \tag{44}$$

As in (17), the oscillating component of the skin-friction is given by

$$\frac{\tau}{\rho b^2} = \left(\frac{c\nu}{b^2}\right)^{1/2} e^{i\omega t}\chi'(0). \tag{45}$$

The variations with frequency of the real and imaginary parts of $\chi'(0)$, and of the phase-advance angle θ_χ, are illustrated in figure 2. For small values of the frequency, $\theta_\chi = \left(1 - \dfrac{1}{A^2}\right)\dfrac{\omega}{c} = 0{\cdot}3418\,\dfrac{\omega}{c}$, indicating again a constant time of anticipation $\theta_\chi/\omega = 0{\cdot}3418/c$. At higher frequencies, θ_χ rises steadily towards the limiting value $\frac{1}{4}\pi$.

Thus, in each of the cases considered here, the behaviour is qualitatively similar to that found by Lighthill for the case in which the stream velocity fluctuates in magnitude. For the Hiemenz layer, which corresponds to the plate making oscillations in the y-direction, the time of anticipation at low frequency was determined as $0{\cdot}18/c$, considerably less than in either of the present cases. Lighthill's analysis was restricted to infinitesimal

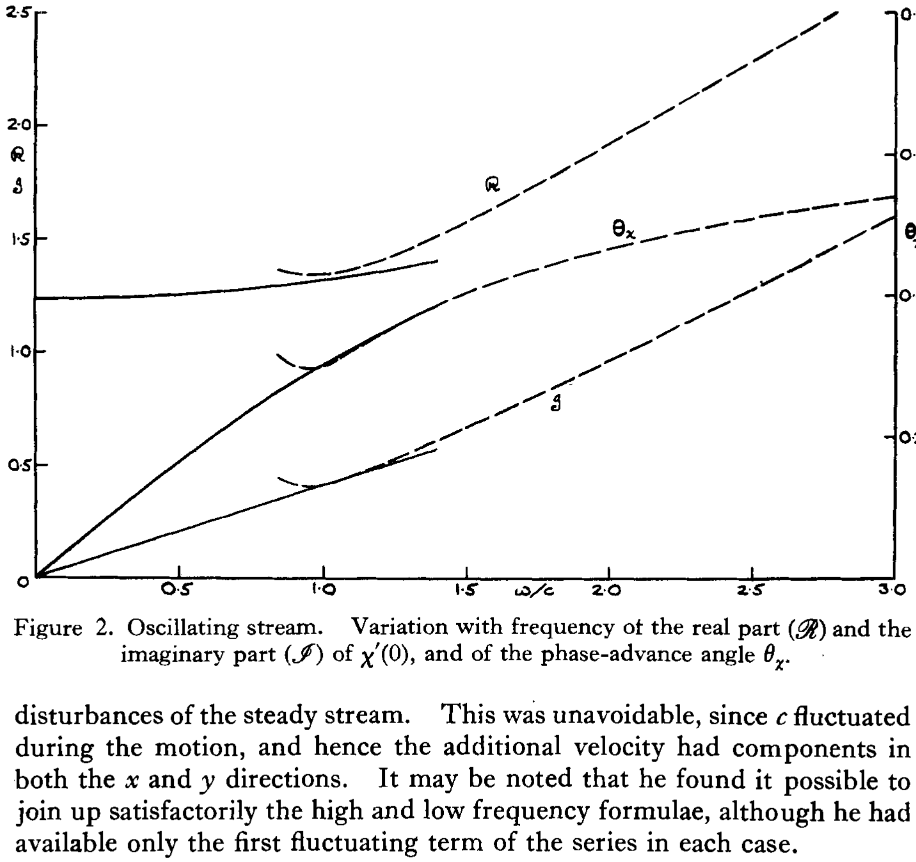

Figure 2. Oscillating stream. Variation with frequency of the real part ($\mathscr{R}$) and the imaginary part ($\mathscr{I}$) of $\chi'(0)$, and of the phase-advance angle θ_χ.

disturbances of the steady stream. This was unavoidable, since c fluctuated during the motion, and hence the additional velocity had components in both the x and y directions. It may be noted that he found it possible to join up satisfactorily the high and low frequency formulae, although he had available only the first fluctuating term of the series in each case.

Finally, a word may be said on the question of fluctuations in heat transfer from a heated plate. This problem was discussed in detail by Lighthill for his case. For the Hiemenz layer, the temperature T is a function of y only, satisfying

$$v\frac{\partial T}{\partial y}=\kappa\frac{\partial^2 T}{\partial y^2}, \quad T=T_0 \quad \text{at } y=0, \quad T\to T_1 \quad \text{as } y\to\infty, \tag{46}$$

where κ is the thermal diffusivity, and T_0 and T_1 are the constant temperatures of the plate and stream. Since, for the problems of this section, the oscillations do not involve any change in v or in the boundary conditions, the steady solution of equation (46) continues to apply unchanged. In other words, the movement of the plate has no effect at all on the temperature distribution and the rate of heat-transfer.

3. Oscillating cylinder

In this section, we consider how the results of § 2 may be applied to a cylinder of arbitrary cross-section.

Any two-dimensional oscillatory motion can be considered as a combination of the following five basic motions:

(i) cylinder fixed, stream oscillates in magnitude;

(ii) cylinder fixed, stream oscillates in direction;

(iii) stream constant, cylinder oscillates in the stream direction;

(iv) stream constant, cylinder oscillates in the transverse direction;

(v) stream constant, cylinder oscillates about its axis.

Here, the word 'stream' is used to denote the fluid velocity at large distances from the cylinder.

Cases (i) and (iii), which are equivalent, were analysed in detail by Lighthill. He assumed that the velocity $U(x)$ outside the boundary layer remains unchanged in form, merely changing in scale in proportion to the velocity of the cylinder relative to the stream.

In the general case, the flow outside the boundary layer differs in form from the corresponding steady flow, except possibly for a circular cylinder, and cannot be determined by boundary layer theory alone. A first approximation, valid for low frequencies, would be to assume the external flow to be the quasi-steady flow, i.e. that due to a steady stream given by the instantaneous relative velocity of the cylinder and the distant fluid. Corrections due to discrepancies in the separated region at the rear of the cylinder might be taken into account in a second approximation. If the cylinder is of aerofoil section, with circulation, the question of the fluctuating component of circulation provides another problem. Except at small values of ω/c, vorticity shed at the rear will still be in the neighbourhood of the cylinder after a few periods, and so will have its effect on the velocity distribution. For, if V is a representative velocity and l a representative length of the cylinder, so that $V = cl$, it is seen that the distance travelled in a period is $2\pi lc/\omega$. For small values of ω/c, the quasi-steady approximation would still be appropriate. For large values of ω/c, the best approximation would be to assume that the circulation remains constant, even though the Kutta–Joukowski condition is no longer satisfied. These considerations indicate that Lighthill's treatment is inadequate for an aerofoil with circulation, even if the fluctuation is confined to the magnitude of the oncoming stream.

However, it is always true that the velocity over the surface near the stagnation point can be written as $U = c(x - x_0)$, where c and x_0 have fluctuating components. If these fluctuations are small, the contributions due to the variations of c and x_0 may be added separately to the basic steady flow. The effect of variations in c are covered by Lighthill's analysis, and, with c constant, the analysis of § 2 becomes applicable. As discussed above, the value of x_0 may depend on other than purely quasi-steady considerations. We now consider the cases (ii), (iv) and (v) in turn.

Case (ii) calls for an immediate application of the oscillating stream formulae of § 2. If the stagnation point is at $x = se^{i\omega t}$, x being measured round the cylinder perimeter, and if c is the velocity gradient at $x = 0$ in the undisturbed flow, then, by equation (35), the solution is given by substituting $b = -cs$ in equation (40). This holds whether or not the value of s is that given by the quasi-steady flow at the appropriate incidence. It is perhaps best to consider s as being found by experiment. The extra contribution due to any fluctuation in c is given by Lighthill's formulae. It is to be noted that, to the first order in the amplitude, no such contribution arises in the case of a symmetrical cylinder when the flow oscillates about the axis of symmetry. It is likely that, for nearly circular cylinders, the contribution to the oscillating skin-friction deduced from equation (45) is indeed the major part for the whole cylinder, since over the shoulders of the cylinder, where all resemblance to a linear gradient has gone, the fluctuations in the quasi-steady external flow are much reduced. In any case, the boundary layer is likely to have become turbulent by this stage.

Case (iv) might, at a first glance, be thought to require the application of the oscillating plate analysis of § 2, but this is not so. The essential point is that, whereas there the axes were fixed in space, here it is natural to consider them as fixed in the cylinder. This is equivalent to superposing a uniform though fluctuating velocity on the system, which has no effect on the relative motion, and so case (iv) is immediately reduced to case (ii), with an oncoming stream as given in the new coordinate system. This procedure is identical with that by which case (iii) is related to case (i).

Case (v) may also be reduced to case (ii), by choosing axes fixed in the cylinder. According to the boundary layer approximation, the only effect of rotation, as of curvature, is a small but still negligible pressure change across the boundary layer. The flow $U(x)$ at the edge of the boundary layer will have a different value from that in the corresponding situation in case (ii), but may be found in suitable cases by quasi-steady considerations. For a circular cylinder, an alternative approach is to use fixed axes and to apply directly the oscillating plate analysis of § 2, the notation being suitable as it stands; a more detailed study of this case, with particular attention to a steadily rotating cylinder, will be made in a later paper.

As a final example, we attempt a quantitative estimate of the overall effects on a circular cylinder making small transverse oscillations. We assume that the flow at the edge of the boundary layer is quasi-steady, and can be written in the simple form

$$U = c(x + x_0), \quad |x + x_0| < x_1, \tag{47}$$

where $x = x_0$ is the stagnation point and x_1 is a constant. We assume that the fluctuation effects are negligible outside this region. Typical values for a cylinder of diameter d in a stream V are $c = 3{\cdot}6\, V/d$, $x_1 = 0{\cdot}4d$. If the transverse velocity component is $\beta e^{i\omega t}$, then, on the quasi-steady hypothesis,

$$x_0 = \frac{\beta d}{2V} e^{i\omega t},$$

for small values of β/V.

For $|x+x_0| < x_1$, the velocity in the boundary layer is given by equation (40) as

$$\begin{aligned} u &= cxf' + cx_0\chi \\ &= c(x+x_0)f' + cx_0(\chi - f'). \end{aligned} \tag{48}$$

The first term of (48) is a contribution to the quasi-steady flow round the cylinder, which results in a fluctuating lift force of amplitude $\beta D/V$, where D is the drag force on the cylinder, in phase with the velocity fluctuation. The second term of (48) involves a skin-friction given by

$$\frac{\tau}{\rho\beta^2} = \left(\frac{c\nu}{\beta^2}\right)^{1/2} \{\chi'(0) - A\} e^{i\omega t}, \tag{49}$$

and the values of this expression for all values of ω are given in § 2. Now

$$\chi'(0) - A = 0{\cdot}4213 \frac{i\omega}{c} + O\left(\frac{\omega^2}{c^2}\right), \tag{50}$$

and thus, for small values of ω/c, the skin-friction acting over the region $|x+x_0| < x_1$ gives rise to a fluctuating torque about the cylinder axis, in phase with the acceleration, of amplitude per unit span T given by

$$T = 0{\cdot}4213\, \rho\beta\omega x_1 d(\nu/c)^{1/2}. \tag{51}$$

For the typical values of c and x_1 given above, and for the particular values $\omega = 0{\cdot}5c$, $\beta = 0{\cdot}1V$, (51) becomes

$$T = 0{\cdot}016\left(\frac{\nu}{Vd}\right)^{1/2} \rho V^2 d^2. \tag{52}$$

Since the fluctuations in velocity for $|x+x_0| \geqslant x_1$, assumed zero here, will in fact be much smaller than those for $|x+x_0| < x_1$, it seems likely that (51) is indeed a reasonable estimate of the torque on the cylinder due to skin-friction.

For the largest values of ω/c, $\chi'(0) - A \sim (i\omega/c)^{1/2}$, but before this regime is reached the basic theory will in many practical cases have broken down, as the acoustic wavelength will have become comparable with the cylinder diameter.

Appendix

Equation (13) of § 2 was

$$\phi_n'' + f\phi_n' - f'\phi_n = \phi_{n-1}; \quad \phi_n(0) = 0, \quad \phi_n(\infty) = 0 \quad (n \geqslant 1). \tag{53}$$

The integrals of the complementary function are f'' and $f''I$, and it is clear that $I = \int_0^\eta e^{-f^0}/f''^2\, d\eta$ diverges as $\eta \to \infty$. To investigate convergence and to obtain numerical coefficients, it will be necessary to consider the behaviour of I in some detail.

The integration of equation (4) has shown that as $\eta \to \infty$,

$$f \sim \xi = \eta - 0{\cdot}6479, \qquad f^0 \sim \tfrac{1}{2}\xi^2 + 0{\cdot}1496.$$

Hence

$$e^{-f^0} \sim Be^{\frac{1}{2}\xi^2},$$

where $B = 0{\cdot}8610$,

Writing $f = \xi + g$ in equation (4), and ignoring terms in g^2, we obtain

$$g''' + \xi g'' - 2g' = 0. \tag{54}$$

Two differentations give $g^{\text{v}} + \xi g^{\text{iv}} = 0$, and hence $g^{\text{iv}} = Ce^{-\frac{1}{2}\xi^2}$, for some constant C. Integrating, we have

$$g''' = -C\int_{\xi}^{\infty} e^{-\frac{1}{2}x^2}\, dx, \tag{55}$$

the well-known error function of which tables are available. Now $g''' = f'''$, and use of the tabulated values of f, together with equation (4) itself, gives $C = 0{\cdot}6451$.

As $\xi \to \infty$, $$f^{\text{iv}} \sim Ce^{-\frac{1}{2}\xi^2},$$

and so $$f''' \sim -\frac{C}{\xi}\, e^{-\frac{1}{2}\xi^2}, \quad f'' \sim \frac{C}{\xi^2} e^{-\frac{1}{2}\xi^2}.$$

Hence $$\left.\begin{aligned} I &\sim \frac{B}{C^2}\int \xi^4 e^{\frac{1}{2}\xi^2}\, d\xi \sim \frac{B}{C^2}\xi^3 e^{\frac{1}{2}\xi^2}, \\ \text{and} \qquad f''I &\sim \frac{B}{C}\xi. \end{aligned}\right\} \tag{56}$$

Returning to equation (47), where ϕ_{n-1} is supposed known, we obtain a solution of the form

$$\phi_n = h_n f'' + k_n f''I, \tag{57}$$

provided that

$$h_n' = -f'' e^{f_0} I \phi_{n-1}, \quad k_n' = f'' e^{f_0} \phi_{n-1}.$$

In view of (56), the boundary conditions on ϕ_n require

$$h_n = \int_0^{\eta} h_n\, d\eta, \qquad k_n = -\int_{\eta}^{\infty} k_n'\, d\eta. \tag{58}$$

Thus ϕ_n has the form already given in equation (14).

It remains to be verified that ϕ_n does in fact tend to zero at infinity; indeed from general boundary layer experience we would expect ϕ_n to tend to zero exponentially. This is the case, as is shown by the following induction argument.

Suppose that for some n, $\phi_{n-1} = O\,(e^{-\frac{1}{2}\xi^2})$ for large ξ, where $\xi = \eta - 0{\cdot}6479$ as before, and here and in what follows $O\,(e^{-p\xi^2})$ is taken to include $O(\xi^q e^{-p\xi^2})$. Then, as a consequence of the results obtained above, $h_n' = O(1)$, $k_n' = O(e^{-\frac{1}{2}\xi^2})$, and so by (57) and (58), $\phi_n = O(e^{-\frac{1}{2}\xi^2})$. Since $\phi_0 = \dfrac{1}{A} f'' = O(e^{-\frac{1}{2}\xi^2})$, the result is true for all n by induction.

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